Deep learning

2.1. Loss and risk

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There are multiple types of inference that we can roughly split into three categories:

- Classification (e.g. object recognition, cancer detection, speech processing),
- regression (e.g. customer satisfaction, stock prediction, epidemiology), and
- density estimation (e.g. outlier detection, data visualization, sampling/synthesis).

The standard formalization for classification and regression considers a measure of probability

$$\mu_{X,Y}$$

over the observation/value of interest, and i.i.d. training samples

$$(x_n, y_n), \; n = 1, \dots, N,$$

and for density estimation

$$\mu_X$$

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So the conditional distribution

$$\mu_{X|Y=y}$$

stands for the distribution of the observable signal for the class y (e.g. "sound of an /ē/", "image of a cat").

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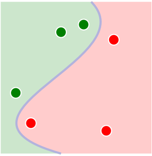
In the simple cases

$$Y = f(X) + \epsilon$$

where f is the deterministic dependency between x and y (e.g. affine), and ϵ is a random noise, independent of X (e.g. Gaussian).

With such a probabilistic perspective, we can more precisely define the three types of inferences we introduced before:



Classification,

- (X, Y) random variables on $\mathscr{Z} = \mathbb{R}^D \times \{1, \dots, C\}$,
- we want to estimate $\text{argmax}_y\, P(Y = y \mid X = x)$.

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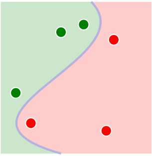

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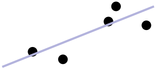

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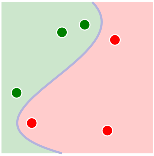

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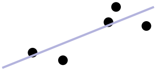
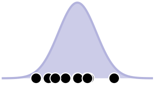


Density estimation,

- X random variable on $\mathscr{Z} = \mathbb{R}^D$,
- we want to estimate μ_X.



The boundaries between these categories are fuzzy:

- Regression allows to do classification through class scores.
- Density models allow to do classification thanks to Bayes' law.

etc.

Risk, empirical risk

Learning consists of finding in a set $\mathscr{F}$ of functionals a "good" f^* (or its parameters' values) usually defined through a loss

$$\ell : \mathscr{F} \times \mathscr{Z} \to \mathbb{R}$$

such that $\ell(f, z)$ increases with how wrong f is on z.

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- for classification:
$$\ell(f, (x, y)) = \mathbf{1}_{\{f(x) \neq y\}},$$
- for regression:
$$\ell(f, (x, y)) = (f(x) - y)^2,$$
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The loss may include additional terms related to f itself.

We are looking for an f with a small **expected risk**

$$R(f) = \mathbb{E}_Z\left(\ell(f, Z)\right),$$

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Although this quantity is unknown, if we have i.i.d. training samples

$$\mathscr{D} = \left\{Z_1, \dots, Z_N\right\},$$

we can compute an estimate, the **empirical risk**:

$$\hat{R}(f; \mathscr{D}) = \hat{\mathbb{E}}_{\mathscr{D}}(\ell(f, Z)) = \frac{1}{N} \sum_{n=1}^{N} \ell(f, Z_n).$$

We have

$$\mathbb{E}_{Z_1,\dots,Z_N}\left(\hat{R}(f;\mathscr{D})\right) \quad = \quad \mathbb{E}_{Z_1,\dots,Z_N}\left(\frac{1}{N}\sum_{n=1}^{N}\ell(f,Z_n)\right)$$

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The empirical risk is an **unbiased estimator** of the expected risk.

Finally, given $\mathscr{D}$, $\mathscr{F}$, and ℓ, "learning" aims at computing

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  Yes if f is not chosen using $\mathscr{D}$. Since the Z_n are independent, we just need to take into account the variance of $\hat{R}(f; \mathscr{D})$.

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  For instance if $|\mathscr{F}| = 1$, we can!

Note that in practice, we call "loss" both the functional

$$\ell : \mathscr{F} \times \mathscr{Z} \to \mathbb{R}$$

and the empirical risk minimized during training

$$\mathscr{L}(f) = \frac{1}{N} \sum_{n=1}^{N} \ell(f, z_n).$$

The end